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A subalgebra of 0-Hecke algebra

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ABSTRACT

Let (W, I) be a finite Coxeter group. In the case where W is a Weyl group, Berenstein and Kazhdan in [A. Berenstein, D. Kazhdan, Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases, in: Quantum Groups, in: Contemp. Math., vol. 433, Amer. Math. Soc., Providence, RI, 2007, pp. 13–88] constructed a monoid structure on the set of all subsets of I using unipotent χ -linear bicrystals. In this paper, we will generalize this result to all types of finite Coxeter groups (including non-crystallographic types). Our approach is more elementary, based on some combinatorics of Coxeter groups. Moreover, we will calculate this monoid structure explicitly for each type.

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1.1. Let W be a Coxeter group generated by the simply reflections s_i (for $i \in I$). Let H be the Iwahori–Hecke algebra associated to W with parameter $q = 0$, i.e., H is a \mathbb{Q} -algebra generated by T_{s_i} for $s_i \in I$ with relations $T_{s_i}^2 = -T_{s_i}$ and the braid relations. The algebra H is called 0-Hecke algebra. It was introduced by Norton in [No]. Representations of H were later studied in the work of Carter [Ca], Hivert, Novelli and Thibon [HNT] and etc. More recently, Stembridge [St] used the 0-Hecke algebra to obtain a new proof for the Möbius function of the Bruhat order of W .

1.2. Set $T'_{s_i} = -T_{s_i}$. For $w \in W$, we define $T'_w = T'_{s_{i_1}} \cdots T'_{s_{i_k}}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of w . Tits's theorem implies that T'_w is well defined. Moreover, we have a binary operation $*$: $W \times W \rightarrow W$ such that $T'_x T'_y = T'_{x*y}$ for any $x, y \in W$. It is easy to see that $(W, *)$ is a monoid with unit element 1.

Now we state our main theorem.

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Theorem 1. Let W be a finite Coxeter group. For any subset $J \subset I$, let w_0^J be the maximal element in the subgroup generated by s_j (for $j \in J$). Then $\{w_0^J w_0^I; J \subset I\}$ is a commutative submonoid of $(W, *)$. In other words, there exists a commutative monoid structure \star_I on the set of subsets of I , such that

$$T'_{w_0^{J_1} w_0^I} T'_{w_0^{J_2} w_0^I} = T'_{w_0^{J_1 \star_I J_2} w_0^I}.$$

Remark. In the case where W is a Weyl group, this result was discovered by Berenstein and Kazhdan in [BK, Proposition 2.30]. Their approach was based on unipotent χ -linear bicrystals. The proof below is more elementary. It is based on some combinatorial properties of Coxeter groups. In the end, we will calculate the operator \star_I explicitly for each type.

Below we introduce a binary operation $\triangleright : W \times W \rightarrow W$ and study some properties about the operations $*$ and \triangleright . In the case where W is a finite Weyl group, these properties were proved in [HL] using geometry of flag varieties.

1.3. We denote by l the length function and \leq the Bruhat order on W .

By [He, Lemma 1.4(1)], for $x, y \in W$, the subset $\{uy; u \leq x\}$ of W contains a unique minimal element which we denote by $x \triangleright y$. Moreover, $x \triangleright y = u'y$ for some $u' \leq x$ and $l(x \triangleright y) = l(y) - l(u')$. We also have that if $s_i x > x$, then $(s_i x) \triangleright y = \min\{s_i(x \triangleright y), x \triangleright y\}$.

There is a similar description for $x * y$.

Lemma 1. Let $x, y \in W$. Then the subset $\{uv; u \leq x, v \leq y\}$ contains a unique maximal element, which equals $x * y$. Moreover, $x * y = u'y = xv'$ for some $u' \leq x$ and $v' \leq y$ and $l(x * y) = l(u') + l(y) = l(x) + l(v')$.

Remark. A slightly weaker version was proved in [He, Lemma 1.4 (2)]. The proof here is similar to [He].

By definition, $s_i * w = \max\{w, s_i w\}$. Now for $a, b \in W$ and $i \in I$ with $s_i a > a$, we have that

$$T'_{(s_i a) * b} = T'_{s_i a} T'_b = T'_{s_i} T'_a T'_b = T'_{s_i} T'_{a * b}.$$

Hence $(s_i a) * b = \max\{a * b, s_i(a * b)\}$ if $s_i a > a$.

We only prove that $\{uv; u \leq x, v \leq y\}$ contains a unique maximal element which equals $x * y$ and $x * y = u'y$ for some $u' \leq x$ with $l(x * y) = l(u') + l(y)$.

We argue by induction on $l(x)$. For $l(x) = 0$, the statement is clear. Assume that $l(x) > 0$ and that the statement holds for all x' with $l(x') < l(x)$. Then there exists $i \in I$ such that $s_i x < x$. By induction hypothesis, the subset $\{uv; u \leq s_i x, v \leq y\}$ contains a unique maximal element $(s_i x) * y$ and there exists $u_1 \leq s_i x$ such that $(s_i x) * y = u_1 y$ and $l(u_1 y) = l(u_1) + l(y)$.

Set $z = x * y$. Then $z = \max\{u_1 y, s_i u_1 y\}$ and $zy^{-1} = u_1$ or $s_i u_1$. In either case, we have that $zy^{-1} \leq x$. If $zy^{-1} = u_1$, then we already know that $l(z) = l(zy^{-1}) + l(y)$. If $zy^{-1} = s_i u_1$, then $s_i u_1 y > u_1 y$ and

$$l(s_i u_1 y) = l(u_1 y) + 1 = l(u_1) + l(y) + 1 \geq l(s_i u_1) + l(y) \geq l(s_i u_1 y).$$

Thus we must have that $l(s_i u_1) = l(u_1) + 1$ and $l(z) = l(zy^{-1}) + l(y)$.

Now for any $u \leq x$ and $v \leq y$. By [Lu, Corollary 2.5], $u \leq s_i x$ or $s_i u \leq s_i x$. By the definition of $(s_i x) * y$, we have that uv or $s_i uv$ is less than or equal to $(s_i x) * y = u_1 y \leq z$. By [Lu, Corollary 2.5], $uv \leq z$. Therefore z is the unique maximal element in the subset $\{uv; u \leq x, v \leq y\}$. \square

Corollary 1. Let $x' \leq x$ and $y' \leq y$. Then $x' * y' \leq x * y$.

Lemma 2. Let $x' \geq x$ and $y' \leq y$, then $x' \triangleright y' \leq x \triangleright y$.

By definition, $x' \triangleright y' \leq x \triangleright y'$. Now we prove that $x \triangleright y' \leq x \triangleright y$ by induction on $l(x)$.

For $l(x) = 0$, $x \triangleright y' = y' \leq y = x \triangleright y$. Now assume that $l(x) > 0$ and that $x_1 \triangleright y' \leq x_1 \triangleright y$ for any x_1 with $l(x_1) < l(x)$. Then there exists $i \in I$ such that $s_i x < x$. By induction hypothesis, $(s_i x) \triangleright y' \leq (s_i x) \triangleright y$.

By [Lu, Corollary 2.5], $x \triangleright y' = \min\{(s_i x) \triangleright y', s_i((s_i x) \triangleright y')\} \leq (s_i x) \triangleright y, s_i((s_i x) \triangleright y)$. Hence $x \triangleright y' \leq \min\{(s_i x) \triangleright y, s_i((s_i x) \triangleright y)\} = x \triangleright y$. The statement is proved. \square

Lemma 3. The action $(W, *) \times W \rightarrow W, (x, y) \mapsto x \triangleright y$ is a left action of the monoid $(W, *)$.

By definition, $1 \triangleright x = x$ for any $x \in W$.

Let $x, y, z \in W$. Then there exists $u \leq x$ and $v \leq y$ such that $y \triangleright z = vz$ and $x \triangleright (vz) = uvz$. By definition $uv \leq x * y$. So $(x * y) \triangleright z \leq x \triangleright (y \triangleright z)$.

On the other hand, there exists $w \leq x$ such that $x * y = wy$ and $l(wy) = l(w) + l(y)$. Then there exists $w' \leq wy$ such that $(wy) \triangleright z = w'z$. Since $l(wy) = l(w) + l(y)$, we may write w' as $w' = w_1 y_1$ for some $w_1 \leq w$ and $y_1 \leq y$. Thus $(x * y) \triangleright z = w_1(y_1 z)$. Notice that $w_1 \leq x$ and $y_1 \leq y$ and $y_1 z \geq y \triangleright z$. By the previous lemma, $x \triangleright (y \triangleright z) \leq w_1(y_1 z) = (x * y) \triangleright z$. The lemma is proved. \square

Lemma 4. Assume that W is finite. Then $(x \triangleright y)w_0^l = x * (yw_0^l)$.

By definition, $x \triangleright y$ is the unique minimal element in $\{uy; u \leq x\}$. Hence $(x \triangleright y)w_0^l$ is the unique maximal element in $\{uyw_0; u \leq x\}$. By Lemma 1, this unique maximal element is $x * (yw_0^l)$. \square

Lemma 5. Let $J_1, J_2 \subset I$. If $(w_0^{J_1} w_0^l) * (w_0^{J_2} w_0^l) = w_0^{J_3} w_0^l$ for some $J_3 \subset I$, then $(w_0^{J_2} w_0^l) * (w_0^{J_1} w_0^l) = w_0^{J_3} w_0^l$. In other words, if $(w_0^{J_1} w_0^l) \triangleright w_0^{J_2} w_0^l = w_0^{J_3} w_0^l$ for some $J_3 \subset I$, then $(w_0^{J_2} w_0^l) \triangleright w_0^{J_1} w_0^l = w_0^{J_3} w_0^l$.

This was proved in [BK, Proposition 2.30 (c)] by applying the anti-automorphism $w \mapsto w_0^l w^{-1} w_0^l$ on W .

Lemma 6. If $J_2 = K \sqcup K'$ with $s_k s_{k'} = s_{k'} s_k$ for any $k \in K$ and $k' \in K'$. Then for any $J_1 \subset I$, we have that

$$(w_0^{J_1} w_0^l) \triangleright w_0^{J_2} = ((w_0^{J_1} w_0^l) \triangleright w_0^K)((w_0^{J_1} w_0^l) \triangleright w_0^{K'}).$$

We fix a reduced expression $w_0^{J_1} w_0^l = s_{i_1} s_{i_2} \cdots s_{i_n}$ for $i_1, i_2, \dots, i_n \in I$. Assume that $v \leq w_0^{J_1} w_0^l$ with $(w_0^{J_1} w_0^l) \triangleright w_0^{J_2} = v w_0^{J_2}$. Then $v \leq w_0^{J_2}$. Hence $v \in W_{J_2}$. Let $v = s_{i_{l_1}} \cdots s_{i_{l_k}}$ be a reduced subexpression. Then $l_1, \dots, l_k \in J_2 = K \cup K'$. By assumption on K and K' , $v = v_1 v_2$ for $v_1 \in W_K$ and $v_2 \in W_{K'}$. So $v w_0^{J_2} = v_1 v_2 w_0^K w_0^{K'} = (v_1 w_0^K)(v_2 w_0^{K'}) \geq ((w_0^{J_1} w_0^l) \triangleright w_0^K)((w_0^{J_1} w_0^l) \triangleright w_0^{K'})$.

On the other hand, assume that $v_1, v_2 \leq w_0^{J_1} w_0^l$ with $(w_0^{J_1} w_0^l) \triangleright w_0^K = v_1 w_0^K$ and $(w_0^{J_1} w_0^l) \triangleright w_0^{K'} = v_2 w_0^{K'}$. Then $v_1 \in W_K$ and $v_2 \in W_{K'}$ and there exist $1 \leq t_1 < t_2 < \cdots < t_u \leq n$ and $1 \leq t'_1 < t'_2 < \cdots < t'_{u'} \leq n$ such that $v_1 = s_{i_{t_1}} \cdots s_{i_{t_u}}$ is a reduced subexpression of v_1 and $v_2 = s_{i_{t'_1}} \cdots s_{i_{t'_{u'}}}$ is a reduced subexpression of v_2 .

Since $K \cap K' = \emptyset$, $\{t_1, \dots, t_u\}$ and $\{t'_1, \dots, t'_{u'}\}$ are disjoint subsets of $\{1, \dots, n\}$. Let v be the element that corresponds to the subexpression $\{t_1, \dots, t_u\} \sqcup \{t'_1, \dots, t'_{u'}\}$. Then it is easy to see that $v = v_1 v_2$. Hence $((w_0^{J_1} w_0^l) \triangleright w_0^K)((w_0^{J_1} w_0^l) \triangleright w_0^{K'}) = (v_1 w_0^K)(v_2 w_0^{K'}) = v_1 v_2 w_0^K w_0^{K'} = v w_0^{J_2} \leq (w_0^{J_1} w_0^l) \triangleright w_0^{J_2}$. \square

Lemma 7. Let $J_1, J_2 \subset I$. Then for any $J_1 \subset J'_1$, we have that

$$(w_0^{J_1} w_0^l) \triangleright w_0^{J_2} = (w_0^{J_1} w_0^{J'_1}) \triangleright ((w_0^{J'_1} w_0^l) \triangleright w_0^{J_2}).$$

Notice that $w_0^{J_1} w_0^I = (w_0^{J_1} w_0^{J'_1})(w_0^{J_2} w_0^I) = (w_0^{J_1} w_0^{J'_1}) * (w_0^{J_2} w_0^I)$. The lemma follows from Lemma 3. \square

Below is the key lemma.

Lemma 8. Assume that W is a irreducible finite Coxeter group and $i, i' \in I$ are end points of the Coxeter graph of W . Let $J_1 = I - \{i\}$ and $J_2 = I - \{i'\}$. Then $(w_0^{J_1} w_0^I) \triangleright w_0^{J_2} = w_0^{J_3}$ for some $J_3 \subset J_1 \cap J_2$.

We will prove this lemma in subsection 1.5. The proof is based on a case-by-case checking. We will also use the result to give an explicit description of the operator \star_I for each type.

Before proving the lemma, we will show that the key lemma implies the main theorem.

1.4. Proof of Theorem 1

By Lemma 5, if $T' w_0^{J_1} w_0^I T' w_0^{J_2} w_0^I = T' w_0^{J_3} w_0^I$, then $T' w_0^{J_2} w_0^I T' w_0^{J_1} w_0^I = T' w_0^{J_3} w_0^I$. Using Lemma 4, we may reformulate the main theorem as follows:

For any $J_1, J_2 \subset I$, we have that $(w_0^{J_1} w_0^I) \triangleright w_0^{J_2} = w_0^{J_3}$ for some $J_3 \subset I$.

We argue by induction on the cardinality of I . By Lemma 6 and Lemma 7, it suffices to prove the case where W is irreducible and J_1, J_2 are connected in the Coxeter graph.

It is easy to see that $(w_0^{J_1} w_0^I) \triangleright w_0^I = (w_0^{J_1} w_0^I) \triangleright w_0^I = w_0^I$ and $(w_0^{J_2} w_0^I) \triangleright w_0^I = (w_0^{J_2} w_0^I) \triangleright w_0^I = 1 = w_0^\emptyset$. Now assume that J_1, J_2 are proper connected subgraph in the Coxeter graph. Then there exists end points $i, i' \in I$ such that $i \notin J_1$ and $i' \notin J_2$. Set $J'_1 = I - \{i\}$ and $J'_2 = I - \{i'\}$. Then $J_1 \subset J'_1$ and $J_2 \subset J'_2$.

By Lemma 7, $(w_0^{J_2} w_0^I) \triangleright w_0^{J'_1} = (w_0^{J_2} w_0^{J'_2}) \triangleright ((w_0^{J'_2} w_0^I) \triangleright w_0^{J'_1})$. By Lemma 8, $(w_0^{J'_2} w_0^I) \triangleright w_0^{J'_1} = w_0^{J_3}$ for some $J_3 \subset J'_1 \cap J'_2$. By induction hypothesis on $W_{J'_2}$, we have that $(w_0^{J_2} w_0^{J'_2}) \triangleright w_0^{J_3} = w_0^{J_4}$ for some $J_4 \subset J'_1 \cap J_2$. By Lemma 5, $(w_0^{J'_1} w_0^I) \triangleright w_0^{J_2} = w_0^{J_4}$.

Again by Lemma 7, $(w_0^{J_1} w_0^I) \triangleright w_0^{J_2} = (w_0^{J_1} w_0^{J'_1}) \triangleright ((w_0^{J'_1} w_0^I) \triangleright w_0^{J_2}) = (w_0^{J_1} w_0^{J'_1}) \triangleright w_0^{J_4}$. By induction hypothesis on $W_{J'_1}$, we have that $(w_0^{J_1} w_0^{J'_1}) \triangleright w_0^{J_4} = w_0^{J_5}$ for some $J_5 \subset J_1 \cap J_2$.

1.5. Proof of Lemma 8

We use the same labeling of Coxeter graph as in [Bo].

For $1 \leq a, b \leq n$, set

$$s_{[a,b]} = \begin{cases} s_a s_{a-1} \cdots s_b, & \text{if } a \geq b, \\ 1, & \text{otherwise.} \end{cases}$$

Type A_n

We have that $w_0^{I-\{1\}} w_0^I = s_{[n,1]}^{-1}$ and $w_0^{I-\{n\}} w_0^I = s_{[n,1]}$. Hence

$$\begin{aligned} (w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= s_{[n,1]}^{-1} \triangleright w_0^{I-\{1\}} = s_{[n,2]}^{-1} w_0^{I-\{1\}} = w_0^{I-\{1,2\}}, \\ (w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{n\}} &= s_{[n,1]}^{-1} \triangleright w_0^{I-\{n\}} = s_{[n-1,1]}^{-1} w_0^{I-\{n\}} = w_0^{I-\{1,n\}}, \\ (w_0^{I-\{n\}} w_0^I) \triangleright w_0^{I-\{n\}} &= s_{[n,1]} \triangleright w_0^{I-\{n\}} = s_{[n-1,1]} w_0^{I-\{n\}} = w_0^{I-\{n-1,n\}}. \end{aligned}$$

Type B_n

We have that $w_0^{I-\{1\}} w_0^I = s_{[n-1,1]}^{-1} s_{[n,1]}$ and $w_0^{I-\{n\}} w_0^I = s_n s_{[n,n-1]}^{-1} \cdots s_{[n,1]}^{-1}$. Hence

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= (s_{[n-1,1]}^{-1} s_{[n,1]}) \triangleright w_0^{I-\{1\}} = s_{[n-1,2]}^{-1} s_{[n,2]} w_0^{I-\{1\}} = w_0^{I-\{1,2\}}, \\
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{n\}} &= (s_{[n-1,1]}^{-1} s_{[n,1]}) \triangleright w_0^{I-\{n\}} = s_{[n-1,1]}^{-1} \triangleright (s_{[n,1]} \triangleright w_0^{I-\{n\}}) = s_{[n-1,1]}^{-1} \triangleright w_0^{I-\{n-1,n\}} \\
&= w_0^{I-\{1,n-1,n\}}, \\
(w_0^{I-\{n\}} w_0^I) \triangleright w_0^{I-\{n\}} &= (s_n s_{[n,n-1]}^{-1} \cdots s_{[n,1]}^{-1}) \triangleright w_0^{I-\{n\}} = s_{n-1} s_{[n-1,n-2]}^{-1} \cdots s_{[n-1,1]}^{-1} w_0^{I-\{n\}} = 1.
\end{aligned}$$

Type D_n

Set

$$\epsilon = \begin{cases} 1, & \text{if } 2 \nmid n; \\ 0, & \text{if } 2 \mid n. \end{cases}$$

We have that

$$\begin{aligned}
w_0^{I-\{1\}} w_0^I &= s_{[n-2,1]}^{-1} s_{[n,1]}, \\
w_0^{I-\{n-1\}} w_0^I &= s_{n-1} (s_{n-2} s_n) \cdots (s_{[n-2,2]}^{-1} s_{n-\epsilon}) (s_{[n-2,1]}^{-1} s_{n-1+\epsilon}), \\
w_0^{I-\{n\}} w_0^I &= s_n (s_{n-2} s_{n-1}) \cdots (s_{[n-2,2]}^{-1} s_{n-1+\epsilon}) (s_{[n-2,1]}^{-1} s_{n-\epsilon}).
\end{aligned}$$

Hence

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= (s_{[n-2,1]}^{-1} s_{[n,1]}) \triangleright w_0^{I-\{1\}} = s_{[n-2,2]}^{-1} s_{[n,2]} w_0^{I-\{1\}} = w_0^{I-\{1,2\}}, \\
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{n-1\}} &= (s_{[n-2,1]}^{-1} s_{[n,1]}) \triangleright w_0^{I-\{n-1\}} = s_{[n-2,1]}^{-1} s_n s_{[n-2,1]} w_0^{I-\{n-1\}} = w_0^{I-\{1,n-1,n\}}, \\
(w_0^{I-\{n-1\}} w_0^I) \triangleright w_0^{I-\{n\}} &= (s_{n-1} (s_{n-2} s_n) \cdots (s_{[n-2,2]}^{-1} s_{n-\epsilon}) (s_{[n-2,1]}^{-1} s_{n-1+\epsilon})) \triangleright w_0^{I-\{n\}} \\
&= s_{n-1} s_{n-2} \cdots s_{[n-2+\epsilon,2]}^{-1} s_{[n-1-\epsilon,1]}^{-1} w_0^{I-\{n\}} \\
&= \begin{cases} s_1 s_3 \cdots s_{n-2}, & \text{if } 2 \nmid n; \\ s_2 s_4 \cdots s_{n-2}, & \text{if } 2 \mid n, \end{cases} \\
(w_0^{I-\{n\}} w_0^I) \triangleright w_0^{I-\{n\}} &= (s_n (s_{n-2} s_{n-1}) \cdots (s_{[n-2,2]}^{-1} s_{n-1+\epsilon}) (s_{[n-2,1]}^{-1} s_{n-\epsilon})) \triangleright w_0^{I-\{n\}} \\
&= s_{[n-1,n-2]}^{-1} \cdots s_{[n-1-\epsilon,2]}^{-1} s_{[n-2+\epsilon,1]}^{-1} w_0^{I-\{n\}} \\
&= \begin{cases} s_2 s_4 \cdots s_{n-1}, & \text{if } 2 \nmid n; \\ s_1 s_3 \cdots s_{n-1}, & \text{if } 2 \mid n. \end{cases}
\end{aligned}$$

Applying the automorphism $\sigma : W \rightarrow W$ which exchanges s_{n-1} and s_n , we also have that $(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{n\}} = w_0^{I-\{1,n-1,n\}}$ and

$$(w_0^{I-\{n-1\}} w_0^I) \triangleright w_0^{I-\{n-1\}} = \begin{cases} (s_2 s_4 \cdots s_{n-3}) s_n, & \text{if } 2 \nmid n; \\ (s_1 s_3 \cdots s_{n-3}) s_n, & \text{if } 2 \mid n. \end{cases}$$

For type E , set $x = s_4 s_3 s_5 s_4 s_2$.

Type E_6

We have that

$$\begin{aligned}w_0^{I-\{1\}}w_0^I &= s_1s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}, \\w_0^{I-\{2\}}w_0^I &= x^{-1}s_{[6,1]}^{-1}s_{[5,1]}x, \\w_0^{I-\{6\}}w_0^I &= s_{[6,1]}s_{[4,6]}^{-1}s_{[3,5]}^{-1}s_2s_4s_3s_1.\end{aligned}$$

Hence

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{1\}} &= (s_1s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}) \triangleright w_0^{I-\{1\}} = (s_{[6,3]}^{-1}x^{-1}s_{[6,2]}^{-1})w_0^{I-\{1\}} = w_0^{\{2,4,5\}}, \\(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{2\}} &= (s_1s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}) \triangleright w_0^{I-\{2\}} = (s_1s_{[6,3]}^{-1}s_4s_3s_5s_4) \triangleright ((s_1s_{[6,3]}^{-1}) \triangleright w_0^{I-\{2\}}) \\&= (s_1s_{[6,3]}^{-1}s_4s_3s_5s_4) \triangleright w_0^{I-\{1,2\}} = s_{[6,3]}^{-1}s_4s_3s_5s_4w_0^{I-\{1,2\}} = s_4s_6, \\(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{6\}} &= (s_1s_{[6,3]}^{-1}x^{-1}s_{[6,1]}^{-1}) \triangleright w_0^{I-\{6\}} = s_1s_{[5,3]}^{-1}x^{-1}s_{[5,1]}^{-1}w_0^{I-\{6\}} = w_0^{\{3,4,5\}}, \\(w_0^{I-\{2\}}w_0^I) \triangleright w_0^{I-\{2\}} &= (x^{-1}s_{[6,1]}^{-1}s_{[5,1]}x) \triangleright w_0^{I-\{2\}} = (x^{-1}) \triangleright ((s_1s_{[6,3]}^{-1}s_{[5,3]}s_1s_4s_3s_5s_4) \triangleright w_0^{I-\{2\}}) \\&= x^{-1} \triangleright (s_3s_5) = 1.\end{aligned}$$

Applying the nontrivial diagram automorphism, we also have that $(w_0^{I-\{6\}}w_0^I) \triangleright w_0^{I-\{6\}} = w_0^{\{2,3,4\}}$ and $(w_0^{I-\{6\}}w_0^I) \triangleright w_0^{I-\{2\}} = s_1s_4$.

Type E_7

We have that

$$\begin{aligned}w_0^{I-\{1\}}w_0^I &= s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1, \\w_0^{I-\{2\}}w_0^I &= s_2s_4s_3s_1s_{[5,2]}s_{[6,4]}^{-1}s_{[5,1]}xs_{[7,1]}xs_{[6,3]}s_{[7,4]}s_2, \\w_0^{I-\{7\}}w_0^I &= s_{[7,1]}xs_{[6,3]}s_1s_{[7,2]}s_{[7,4]}^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{1\}} &= (s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1) \triangleright w_0^{I-\{1\}} \\&= (s_3s_4s_2s_{[5,3]}) \triangleright ((s_{[6,2]}s_{[6,4]}^{-1}s_{[7,2]}xs_{[6,3]}) \triangleright w_0^{I-\{1\}}) \\&= (s_3s_4s_2s_{[5,3]}) \triangleright w_0^{\{2,4,5,7\}} = s_2s_5s_7, \\(w_0^{I-\{1\}}w_0^I) \triangleright w_0^{I-\{2\}} &= (s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}s_{[7,1]}xs_{[6,3]}s_1) \triangleright w_0^{I-\{2\}} \\&= (s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}) \triangleright ((s_{[7,1]}xs_{[6,3]}s_1) \triangleright w_0^{I-\{2\}}) \\&= (s_1s_3s_4s_2s_{[5,3]}s_1s_{[6,2]}s_{[6,4]}^{-1}) \triangleright (s_1s_4s_{[6,3]}s_1) = 1,\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{7\}} &= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1) \triangleright w_0^{I-\{7\}} \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1}) \triangleright ((s_{[6,1]} x s_{[6,3]} s_1) \triangleright w_0^{I-\{7\}}) \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1}) \triangleright w_0^{I-\{6,7\}} \\
&= s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[5,2]} s_{[5,4]}^{-1} w_0^{I-\{6,7\}} = w_0^{\{3,4,5\}},
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{2\}} w_0^I) \triangleright w_0^{I-\{2\}} &= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]} x s_{[6,3]} s_{[7,4]} s_2) \triangleright w_0^{I-\{2\}} \\
&= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x) \triangleright ((s_{[7,1]} x s_{[6,3]} s_{[7,4]}) \triangleright w_0^{I-\{2\}}) \\
&= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x) \triangleright (s_1 s_4 s_6) = 1,
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{2\}} w_0^I) \triangleright w_0^{I-\{7\}} &= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]} x s_{[6,3]} s_{[7,4]} s_2) \triangleright w_0^{I-\{7\}} \\
&= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1}) \triangleright ((s_{[5,1]} x s_{[6,1]} x s_{[6,3]} s_{[6,4]} s_2) \triangleright w_0^{I-\{7\}}) \\
&= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1}) \triangleright w_0^{\{1,3,4,6\}} = 1,
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{7\}} w_0^I) \triangleright w_0^{I-\{7\}} &= (s_{[7,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[7,4]}^{-1}) \triangleright w_0^{I-\{7\}} = (s_{[6,1]} x s_{[6,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1}) w_0^{I-\{7\}} \\
&= w_0^{\{2,3,4,5\}}.
\end{aligned}$$

Type E_8

We have that

$$\begin{aligned}
w_0^{I-\{1\}} w_0^I &= s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1 s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[6,4]}^{-1} s_{[8,1]} x s_{[6,3]} s_1, \\
w_0^{I-\{2\}} w_0^I &= s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]} x s_{[6,3]} s_{[7,4]} s_2 s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[7,4]}^{-1} s_{[8,1]} x s_{[6,3]} s_{[7,4]} s_2, \\
w_0^{I-\{8\}} w_0^I &= s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[7,4]}^{-1} s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[8,4]}^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1) \\
&\triangleright ((s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[6,4]}^{-1} s_{[8,1]} x s_{[6,3]} s_1) \triangleright w_0^{I-\{1\}}) \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1) \triangleright (s_3 s_5 s_7) \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{2\}} &= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1 s_{[8,1]} x s_{[6,3]} s_1) \\
&\triangleright ((s_{[7,2]} s_{[6,4]}^{-1} s_{[8,1]} x s_{[6,3]} s_1) \triangleright w_0^{I-\{2\}}) \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1 s_{[8,1]} x s_{[6,3]} s_1) \triangleright (s_1 s_3 s_{[6,3]} s_1) \\
&= 1,
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{8\}} &= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1) \\
&\triangleright ((s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[6,4]}^{-1} s_{[8,1]} x s_{[6,3]} s_1) \triangleright w_0^{I-\{8\}}) \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1) \triangleright w_0^{\{2,3,4,5,6\}} \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1}) \triangleright ((s_{[7,1]} x s_{[6,3]} s_1) \triangleright w_0^{\{2,3,4,5,6\}}) \\
&= (s_1 s_3 s_4 s_2 s_{[5,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1}) \triangleright w_0^{\{3,4,5\}} = 1,
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{2\}} w_0^I) \triangleright w_0^{I-\{2\}} &= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]} x s_{[6,3]} s_{[7,4]} s_2 s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[7,4]}^{-1}) \\
&\triangleright ((s_{[8,1]} x s_{[6,3]} s_{[7,4]} s_2) \triangleright w_0^{I-\{2\}}) \\
&= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]} x s_{[6,3]} s_{[7,4]} s_2 s_{[8,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[7,4]}^{-1}) \\
&\triangleright (s_1 s_4 s_6 s_{[7,3]} s_1) = 1,
\end{aligned}$$

$$\begin{aligned}
(w_0^{I-\{2\}} w_0^I) \triangleright w_0^{I-\{8\}} &= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]}) \\
&\triangleright (x s_{[6,3]} s_{[6,4]} s_2 s_{[7,1]} x s_{[6,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_{[7,4]} s_2 w_0^{I-\{8\}}) \\
&= (s_2 s_4 s_3 s_1 s_{[5,2]} s_{[6,4]}^{-1} s_{[5,1]} x s_{[7,1]}) \triangleright w_0^{\{1,5,6\}} = 1,
\end{aligned}$$

$$(w_0^{I-\{8\}} w_0^I) \triangleright w_0^{I-\{8\}} = s_{[6,1]} x s_{[6,3]} s_1 s_{[6,2]} s_{[6,4]}^{-1} s_{[7,1]} x s_{[6,3]} s_1 s_{[7,2]} s_{[7,4]}^{-1} w_0^{I-\{8\}} = w_0^{\{2,3,4,5\}}.$$

Type F_4

We have that $w_0^{I-\{1\}} w_0^I = s_{[4,1]}^{-1} s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_{[4,1]}$ and $w_0^{I-\{4\}} w_0^I = s_{[4,1]} s_3 s_2 s_3 s_4 s_3 s_2 s_3 s_{[4,1]}^{-1}$. Hence

$$\begin{aligned}
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} &= (s_{[4,1]}^{-1} s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_{[4,1]}) \triangleright w_0^{I-\{1\}} = 1, \\
(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{4\}} &= (s_{[4,1]}^{-1} s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_{[4,1]}) \triangleright w_0^{I-\{4\}} = 1.
\end{aligned}$$

Applying the nontrivial diagram automorphism, we also have that $(w_0^{I-\{4\}} w_0^I) \triangleright w_0^{I-\{4\}} = 1$.

Type H_3

We have that $w_0^{I-\{1\}} w_0^I = s_1 s_2 s_1 s_2 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1$ and $w_0^{I-\{3\}} w_0^I = s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3$. Hence it is easy to see that $(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} = (w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{3\}} = (w_0^{I-\{3\}} w_0^I) \triangleright w_0^{I-\{3\}} = 1$.

Type H_4

We have that $w_0^{I-\{1\}} w_0^I = s_1 s_2 s_1 s_2 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 (s_{[4,1]} s_2 s_1 s_{[3,1]} s_2 s_3)^4 s_3 s_2$ and $w_0^{I-\{4\}} w_0^I = (s_{[4,1]} s_2 s_1 s_{[3,1]} s_2 s_3)^4 s_4$. Hence it is easy to see that $(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} = (w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{4\}} = (w_0^{I-\{4\}} w_0^I) \triangleright w_0^{I-\{4\}} = 1$.

Type I_m

It is easy to see that $(w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{1\}} = (w_0^{I-\{1\}} w_0^I) \triangleright w_0^{I-\{2\}} = (w_0^{I-\{2\}} w_0^I) \triangleright w_0^{I-\{2\}} = 1$.

1.6. Now we will calculate the operator \star_I explicitly for each type. This is based on the previous subsection, the equalities $J_1 \star_I J_2 = J_2 \star_I J_1$ (see Lemma 5), $J'_1 \star_I J_2 = J'_1 \star_{J_1} (J_1 \star_I J_2)$ for any $J'_1 \subset J_1$ (see Lemma 7) and the inequality $J_1 \star_I J_2 \subset J'_1 \star_I J'_2$ for $J_1 \subset J'_1$ and $J_2 \subset J'_2$ (see Lemma 2). We just list below the cases where J_1 and J_2 are proper connected subgraph of the Coxeter graph of W .

For the case where J_1 and J_2 are not necessarily connected and J'_1, \dots, J'_l (resp. J''_1, \dots, J''_r) are the connected components of J_1 (resp. J_2), we have that $J_1 \star_I J_2 = \bigsqcup_{1 \leq i \leq l, 1 \leq i' \leq r} (J'_i \star_I J''_{i'})$ (see Lemma 6).

Type A_n : Let $J_1 = \{a, a+1, \dots, n-b\}$, $J_2 = \{a', a'+1, \dots, n-b'\}$. Then

$$J_1 \star_I J_2 = \{a+a'-1, a+a', \dots, n-b-b'\}.$$

Type B_n : Let $J_1 = \{a, a+1, \dots, n-b\}$, $J_2 = \{a', a'+1, \dots, n-b'\}$. Then

$$J_1 \star_I J_2 = \begin{cases} \{a+a'-1, a+a', \dots, n\}, & \text{if } b=b'=0; \\ \{a+a'-1, a+a', \dots, n-b'-1\}, & \text{if } b=0, b' \geq 1; \\ \emptyset, & \text{if } b, b' \geq 1. \end{cases}$$

Type D_n : If $I - J_1 = n-1$ or n and $I - J_2 = n-1$ or n , then $J_1 \star_I J_2$ was already calculated in the previous subsection.

If $\{n-1, n\} \not\subset J_1$ and $\{n-1, n\} \not\subset J_2$ and J_1 or J_2 contains at most $n-2$ elements, then $J_1 \star_I J_2 = \emptyset$.

Otherwise, we may assume that $\{n-1, n\} \subset J_1$, i.e. $J_1 = \{a, a+1, \dots, n\}$ for some a . If $\{n-1, n\} \subset J_2$, i.e. $J_2 = \{a', a'+1, \dots, n\}$ for some a' , then $J_1 \star_I J_2 = \{a+a'-1, a+a', \dots, n\}$.

If $\{n-1, n\} \not\subset J_2$, we may assume without loss of generality that $n \notin J_2$, i.e. $J_2 = \{a', a'+1, \dots, n-b'\}$ for some a', b' with $b' \geq 1$. Then $J_1 \star_I J_2 = \{a+a'-1, a+a', \dots, n-b'-1\}$.

Type E_6 : If $2 \notin J_1$ and $2 \notin J_2$, then $J_1 \star_I J_2 = \emptyset$.

If $2 \in J_1$ and $2 \notin J_2$, then

$$J_1 \star_I J_2 = \begin{cases} \{4, 6\}, & \text{if } J_1 = I - \{1\}, J_2 = I - \{2\}, \\ \{1, 4\}, & \text{if } J_1 = I - \{6\}, J_2 = I - \{2\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If $2 \in J_1 \cap J_2$, then

$$J_1 \star_I J_2 = \begin{cases} \{2, 4, 5\}, & \text{if } J_1 = J_2 = I - \{1\}, \\ \{2, 3, 4\}, & \text{if } J_1 = J_2 = I - \{6\}, \\ \{3, 4, 5\}, & \text{if } \{J_1, J_2\} = \{I - \{1\}, I - \{6\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{1\}, I - \{1, 6\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{6\}, I - \{1, 6\}\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Type E_7 : For proper subsets J_1 and J_2 , we have that

$$J_1 \star_I J_2 = \begin{cases} \{2, 5, 7\}, & \text{if } J_1 = J_2 = I - \{1\}, \\ \{2, 3, 4, 5\}, & \text{if } J_1 = J_2 = I - \{7\}, \\ \{3, 4, 5\}, & \text{if } \{J_1, J_2\} = \{I - \{1\}, I - \{7\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{7\}, I - \{1, 7\}\}, \\ \{4\}, & \text{if } \{J_1, J_2\} = \{I - \{7\}, I - \{6, 7\}\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Type E_8 : For proper subsets J_1 and J_2 , we have that

$$J_1 \star_I J_2 = \begin{cases} \{2, 3, 4, 5\}, & \text{if } J_1 = J_2 = I - \{8\}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Type F , H and I : For proper subsets J_1, J_2 of I , we always have that $J_1 \star_I J_2 = \emptyset$.

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